Invariant types in NIP theories

Roland Walker

University of Illinois at Chicago

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Reference

Simon, Pierre (2015). Invariant types in NIP theories. *Journal of Mathematical Logic* 15 (2):1550006.

Invariant Types

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.

Definition

A global type $p \in S_U(x)$ is *invariant over M* (or M-invariant) iff: for all $\sigma \in \mathrm{Aut}(U/M)$, we have

$$\sigma(p) = p$$

(i.e., $\phi(x, b) \in p$ iff $\phi(x, \sigma(b)) \in p$).

Fact

If $p \in S_U(x)$ is finitely satisfiable in M, then p is invariant over M.

Fact

If $p \in S_U(x)$ is definable over M, then p is invariant over M.

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Note: In a stable context, these three notions coincide.

The (p, q)-Theorem

Let $p \geq q$ be positive integers.

Definition

A set system $(X, \mathcal{S})$ has the (p, q)-*property* iff:

- $\varnothing \notin \mathcal{S}$ and
- out of every p sets in $\mathcal{S}$, some q have nonempty intersection.

(p, q)-Theorem (Alon-Kleitman 1992)

There exists $N = N(p, q)$ such that for all *finite* set systems $(X, \mathcal{S})$ with the (p, q)-property and $\mathrm{VC}^*(\mathcal{S}) < q$,

there is a subset of X with size at most N which intersects every set in $\mathcal{S}$.

Applying the (p, q)-Theorem

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.

Let $\phi(x, y)$, $\psi(y) \in \mathcal{L}_M$ with ϕ NIP.

Proposition 2.5 + Uniform Bound $N = N(p, q)$

Suppose $p \geq q > \mathrm{VC}^*(\phi)$ and

$$\mathcal{S}_{\phi,\psi} := \{\phi(M, b) \,:\, b \in \psi(M)\}$$

has the (p, q)-property.

Then there are finitely many global types $p_0, \ldots, p_{N-1} \in S_U(x)$ where $N = N(p, q)$ such that for each $b \in \psi(U)$, $\phi(x, b)$ is in one of them.

Proof: (p, q)-Theorem . . .

Proof: Let $N = N(p, q)$ be given by the (p, q)-Theorem.

Let

$$\Gamma(x_0, \ldots, x_{N-1}) = \left\{ \bigvee_{i<N} \phi(x_i, b) \,:\, b \in \psi(U) \right\}.$$

Given a finite $\Gamma' \subseteq \Gamma$, we can apply the (p, q)-Theorem to

$$X = M \qquad \text{and} \qquad \mathcal{S} = \{\phi(M, b) \,:\, b \in \Gamma'\}$$

to obtain $a'_0, \ldots, a'_{N-1} \models \Gamma'$.

By compactness, there are $a_0, \ldots, a_{N-1} \models \Gamma$.

Let $p_i = \mathrm{tp}_U(a_i)$. ■

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- We want a model-theoretic argument which does not rely on the (p, q)-theorem.
- Simon provides such an argument if we are willing to give up the uniform bound on N.

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.

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Proposition 2.5 (version 1)

Suppose $p \geq q > \mathrm{VC}^*(\phi)$ and

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has the (p, q)-property.

Then there are finitely many global types $\cancel{p_0, \ldots, p_{N-1} \in S_U(x)}$ ~~where~~ $\cancel{N = N(p, q)}$ such that for each $b \in \psi(U)$, $\phi(x, b)$ is in one of them.

Or equivalently ...

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.

Let $\phi(x, y)$, $\psi(y) \in \mathcal{L}_M$ with ϕ NIP.

Proposition 2.5 (version 2)

~~Suppose~~ $\cancel{p \geq q > \mathrm{VC}^*(\phi)}$ ~~and~~

$$\cancel{\mathcal{S}_{\phi,\psi} := \{\phi(M, b) : b \in \psi(M)\}}$$

~~has the~~ $\cancel{(p, q)}$~~-property.~~

Suppose for all $b \in \psi(U)$, $\phi(x, b)$ does not divide over M.

Then there are finitely many global types $\cancel{p_0, \ldots, p_{N-1} \in S_U(x)}$ ~~where~~ $\cancel{N = N(p, q)}$ such that for each $b \in \psi(U)$, $\phi(x, b)$ is in one of them.

Cleaning things up . . .

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Or equivalently . . .

Proposition 2.5 (version 3)

Suppose $b \in U$ such that $\phi(x, b)$ does not divide over M.

Then for all global coheirs $q \in S_U(y)$ of $\mathrm{tp}_M(b)$, there is $a \in U$ such that $\phi(a, y) \in q$.

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Note that a depends on q.

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.

Let $\phi(x, y) \in \mathcal{L}_M$ be NIP.

Conjecture 2.15 (Chernikov-Simon 2015)

Suppose $b \in U$ such that $\phi(x, b)$ does not divide over M.

Then there is $\psi(y) \in \mathrm{tp}_M(b)$ such that $\{\phi(x, b) : b \in \psi(M)\}$ is consistent.

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.

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Let $\mathcal{L}$ be a language, $\mathcal{M}$ an NIP $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.

Let $\phi(x, y) \in \mathcal{L}_M$.

Theorem 2.17

Suppose $b \in U$ such that $\phi(x, b)$ does not divide over M and $\mathrm{tp}_M(b)$ has only countably many global coheirs.

Then there is $a \in U$ such that for all global coheirs $q \in S_U(y)$ of $\mathrm{tp}_M(b)$, $\phi(a, y) \in q$.

$$
\begin{tikzcd}
\boxed{(p, q)\text{-Theorem}} \arrow[d, Rightarrow] & & \\
\text{Proposition 2.5} + \text{Uniform Bound} \arrow[d, Rightarrow] & & \text{Theorem 2.17} \\
\text{Proposition 2.5} & & \boxed{\text{Conjecture 2.15}} \arrow[ll, Rightarrow] \arrow[u, Rightarrow]
\end{tikzcd}
$$

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\begin{tikzcd}
\boxed{(p,q)\text{-Theorem}} \arrow[d, Rightarrow] & \\
\text{Proposition 2.5} + \text{Uniform Bound} \arrow[d, Rightarrow] \arrow[r, dashed, no head, "\text{How do these compare?}"] & \text{Theorem 2.17} \\
\text{Proposition 2.5} & \boxed{\text{Conjecture 2.15}} \arrow[l, Rightarrow] \arrow[u, Rightarrow]
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Let $\phi(x, y)$, $\psi(y) \in \mathcal{L}_M$ with ϕ NIP.

Proposition 2.5 + Uniform Bound $N = N(p, q)$

Suppose $p \geq q > \mathrm{VC}^*(\phi)$ and $\mathcal{S}_{\phi,\psi} := \{\phi(M, b) : b \in \psi(M)\}$ has the (p, q)-property.

Then there are finitely many global types $p_0, \ldots, p_{N-1} \in S_U(x)$ where $N = N(p, q)$ such that for each $b \in \psi(U)$, $\phi(x, b)$ is in one of them.

Further assume $\mathcal{M}$ is NIP.

Theorem 2.17

Suppose $b \in U$ such that $\phi(x, b)$ does not divide over M and $\mathrm{tp}_M(b)$ has only countably many global coheirs.

Then there is $a \in U$ such that for all global coheirs $q \in S_U(y)$ of $\mathrm{tp}_M(b)$, $\phi(a, y) \in q$.

Note that a does not depend on q.

In an NIP context, restricted Morley sequences determine global types.

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.

Let $p, q \in S_U(x)$ be invariant over M and $\phi(x, y) \in \mathcal{L}_M$ be NIP.

Fact 1.3

If $p^{(\omega)}\restriction_M = q^{(\omega)}\restriction_M$, then $p^\phi = q^\phi$.

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Fact 1.3

If $p^{(\omega)}\restriction_M = q^{(\omega)}\restriction_M$, then $p^\phi = q^\phi$.

For all $i < \omega$, let $a_i \models p\restriction_{Ma_{<i}}$.

Definition

We call $(a_i)_{i<\omega}$ a *Morley sequence* of p over M.

It follows $(a_i)_{i<\omega} \models p^{(\omega)}\restriction_M$.

Convergence

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.

Let $p \in S_U(x)$ be invariant over M and $\phi(x, y) \in \mathcal{L}_M$ be NIP.

Suppose $I := (a_i)_{i<\omega} \models p^{(\omega)}\lfloor_M$ and each $p_i = \mathrm{tp}_U(a_i)$.

Lemma

If $p_i\lfloor_{MI} \to p\lfloor_{MI}$, then $p_i^\phi \to p^\phi$.

Proof: Since $S_U(x)$ is compact, there is an accumulation point $q \in S_U(x)$ of $(p_i)_{i<\omega}$. Since $p_i\lfloor_{MI} \to p\lfloor_{MI}$, we have $q\lfloor_{MI} = p\lfloor_{MI}$.

Thus $I \models q^{(\omega)}\lfloor_M$, so $p^{(\omega)}\lfloor_M = q^{(\omega)}\lfloor_M$. By Fact 1.3, $p^\phi = q^\phi$.

Because q is arbitrary, all such accumulation points must agree with p^ϕ. So $p_i^\phi \to p^\phi$. ■

Dividing

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.

Let $A \subseteq U$, $b \in U$, and $\phi(x, y) \in \mathcal{L}$.

Definition

We say $\phi(x, b)$ *divides over A* iff: there exists an A-indiscernible sequence $(b_i : i < \omega)$ with $b_0 = b$ such that $\{\phi(x, b_i) : i < \omega\}$ is inconsistent.

Let $(b_i : i < \omega) \subseteq U$ be indiscernible with $|b_i| = |y|$.

Suppose $\mathrm{VC}^*(\phi) = d < \omega$.

Lemma 2.2

If $\{\phi(x, b_i) : i < \omega\}$ is $(d + 1)$-consistent, then it is consistent.

Let $\mathcal{L}$ and $\mathcal{M}$ be countable. Let $\mathcal{N} \succ \mathcal{M}$ be $\aleph_1$-saturated.

Suppose $\phi(x, y) \in \mathcal{L}_M$ is NIP, $b \in U$, and $\phi(x, b)$ does not divide over $\mathcal{M}$.

Theorem 2.10'

If $q \in S_U(y)$ is a coheir of $\mathrm{tp}_M(b)$, then there is $a \in N$ with $\phi(a, y) \in q$.

Proof: Let $I \subseteq N$ be such that

$$I := (b'_i)_{i<\omega} \models q^{(\omega)}\restriction_M .$$

Let $(\theta_i(y) : i < \omega)$ enumerate $q\restriction_{MI}$. For $k < \omega$, let $\psi_k(y) = \bigwedge_{i<k} \theta_i(y)$.

Observe that we cannot have $(b_i)_{i<\omega} \subseteq U$ such that

(i) $\{\phi(x, b_i) \nleftrightarrow \phi(x, b_{i+1}) : i < \omega\}$ is satisfiable and

(ii) for all $i < \omega$, we have $b_i \models \psi_i$

since our first lemma and (ii) imply that $\mathrm{tp}^{\phi^*}(b_i) \to q^{\phi^*}$, while (i) precludes convergence.

Let $\pi(x) = \bigwedge_{i \leq \mathrm{VC}^*(\phi)+1} \phi(x, b'_i)$ and

$$\gamma_n(x, y_0, \ldots, y_n) = \bigwedge_{i<n} [\phi(x, y_i) \nleftrightarrow \phi(x, y_{i+1})].$$

By the previous observation and compactness, let $n < \omega$ and $b_0, \ldots, b_{n-1} \in M$ such that

(1) $\mathcal{U} \models \exists x[\pi(x) \wedge \gamma_{n-1}(x, b_0, \ldots, b_{n-1})]$,

(2) for all $i < n$, $b_i \models \psi_i$, and

(3) for all $b_n \in \psi_n(M)$, $\mathcal{U} \models \neg\exists x[\pi(x) \wedge \gamma_n(x, b_0, \ldots, b_n)]$.

Recall:

- $\pi(x) = \bigwedge_{i \leq \mathrm{VC}^*(\phi)+1} \phi(x, b'_i)$
- $\gamma_n(x, y_0, \ldots, y_n) = \bigwedge_{i<n} [\phi(x, y_i) \not\leftrightarrow \phi(x, y_{i+1})]$
- (1) $\mathcal{U} \models \exists x[\pi(x) \wedge \gamma_{n-1}(x, b_0, \ldots, b_{n-1})]$

Let $b'_\omega \in N$ be such that $b'_\omega \models q\restriction_{MI}$.

Claim

There exists $a_* \in N$ satisfying $\pi(x) \wedge \gamma_{n-1}(x, b_0, \ldots, b_{n-1}) \wedge \phi(x, b'_\omega)$.

Proof of Claim: Let $\phi'(x, y) = \phi(x, y) \wedge \gamma_{n-1}(x, b_0, \ldots, b_{n-1})$.

Now (1) implies that $\bigwedge_{i \leq \mathrm{VC}^*(\phi)+1} \phi'(x, b'_i)$ is satisfiable.

Since $\mathrm{VC}^*(\phi') \leq \mathrm{VC}^*(\phi) + 1$, Lemma 2.2 ensures there is an $a_* \in N$ satisfying

$$\{\phi'(x, b'_i) : i \leq \omega\}.$$

$\square$

Let $\gamma(x) = \bigwedge_{i<n} \phi(x, b_i)^{\epsilon_i}$ with each $\epsilon_i < 2$ such that $\mathcal{U} \models \gamma(a_*)$.

By (3), there is $\epsilon_n < 2$ such that for all $b_n \in \psi_n(M)$,

$$\mathcal{U} \models \pi(x) \wedge \gamma(x) \to \phi(x, b_n)^{\epsilon_n}.$$

Since q is finitely satisfiable in $\psi_n(M)$,

$$\pi(x) \wedge \gamma(x) \to \phi(x, y)^{\epsilon_n} \in q(y).$$

Further, since

$$\mathcal{U} \models \pi(a_*) \wedge \gamma(a_*) \wedge \phi(a_*, b'_\omega)$$

and $b'_\omega \models q\restriction_{MI}$, we must have $\epsilon_n = 1$.

Thus, for all $b_n \in \psi_n(M)$, $\mathcal{U} \models \phi(a_*, b_n)$.

Finally, since q is finitely satisfiable in $\psi_n(M)$, we conclude that $\phi(a_*, y) \in q$. ■

So far we have proved Proposition 2.5 in a countable context . . .

Let $\mathcal{L}$ and $\mathcal{M}$ be countable. Let $\mathcal{U} \succ \mathcal{M}$ be a monster model.

Let $\phi(x, y) \in \mathcal{L}_M$ be NIP.

Proposition 2.5' (version 3)

Suppose $b \in U$ such that $\phi(x, b)$ does not divide over M.

Then for all global coheirs $q \in S_U(y)$ of $\mathrm{tp}_M(b)$, there is $a \in U$ such that $\phi(a, y) \in q$.

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Or equivalently, if $\psi(y) \in \mathcal{L}_M$ . . .

Proposition 2.5' (version 2)

Suppose for all $b \in \psi(U)$, $\phi(x, b)$ does not divide over M.

Then there are finitely many global types such that for each $b \in \psi(U)$, $\phi(x, b)$ is in one of them.

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.

Let $\phi(x, y), \psi(y) \in \mathcal{L}_M$ with ϕ NIP.

Lemma

Suppose for all $b \in \psi(U)$, $\phi(x, b)$ does not divide over M.

Then there exist $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathcal{M}_0 \prec \mathcal{M}$, both countable, such that

- $\phi, \psi \in \mathcal{L}'_{M_0}$ and
- for all $b \in \psi(U)$, $\phi(x, b)$ does not divide over $\mathcal{M}'_0$ ($\mathcal{L}'$-reduct of $\mathcal{M}_0$).

Proof:

Let $\Sigma(y_i : i < \omega) = \mathrm{Th}_M(\mathcal{M}) + (y_i)_{i<\omega}$ indisc. over $M + \psi(y_0)$.

Let $d = \mathrm{VC}^*(\phi)$ and $\theta(y_0, \ldots, y_d) = \exists x \bigwedge_{i \leq d} \phi(x, y_i)$.

Since $\Sigma \vdash \theta$, there is a finite $\Gamma(y_0, \ldots, y_{n-1}) \subseteq \Sigma$ such that $\Gamma \vdash \theta$.

Choose a finite sublanguage $\mathcal{L}' \subseteq \mathcal{L}$ and a countable model $\mathcal{M}_0 \prec \mathcal{M}$ such that $\Gamma \subseteq \mathcal{L}'_{M_0}$ and $\theta, \psi \in \mathcal{L}'_{M_0}$.

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.

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Proof (cont.): Suppose $(b_i)_{i<\omega} \subseteq U = U'$ is indiscernible over $\mathcal{M}'_0$ with $b_0 \in \psi(U)$. By the Coincidence Lemma and the indiscernibility of $(b_i)_{i<\omega}$,

$$\mathcal{U}' \models \Gamma(b_0, \ldots, b_{n-1}), \qquad \text{so} \qquad \mathcal{U}' \models \theta(b_0, \ldots, b_d).$$

Further $\mathrm{VC}^*_{\mathcal{U}}(\phi) = \mathrm{VC}^*_{\mathcal{U}'}(\phi)$. So by Lemma 2.2, $\{\phi(x, b_i) : i < \omega\}$ is satisfiable in $\mathcal{U}'$. Thus $\phi(x, b)$ does not divide over $\mathcal{M}'_0$. ∎

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.

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Proposition 2.5 (version 2)

Suppose for all $b \in \psi(U)$, $\phi(x, b)$ does not divide over M.

Then there are finitely many global types such that for each $b \in \psi(U)$, $\phi(x, b)$ is in one of them.

Proof: By the Lemma, there exist $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathcal{M}_0 \prec \mathcal{M}$, both countable, such that for all $b \in \psi(U)$, $\phi(x, b)$ does not divide over $\mathcal{M}_0'$.

By Proposition 2.5' there exist $p_0', \ldots, p_{n-1}' \in S_{U'}(x)$ such that for each $b \in \psi(U)$, $\phi(x, b)$ is in one of them.

For each i, let $p_i \in S_U(x)$ extend p_i'. ∎

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For each i, let $p_i \in S_U(x)$ extend p_i'. ■

This result has interesting topological consequences for type space ...

Let $\mathcal{M} \prec^+ \mathcal{N}$. Fix $\phi(x, y) \in \mathcal{L}_M$ NIP. Let $p \in S_U^{\phi}(x)$ be invariant over M.

Define the function

$$d_p^{\phi} : S_M(y) \to 2$$

such that for all $t \in S_M(y)$,

$$p \vdash \phi(x, t)^{d_p^{\phi}(t)}.$$

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such that for all $t \in S_M(y)$,

$$p \vdash \phi(x, t)^{d_p(t)}.$$

Let $\mathcal{M} \prec^+ \mathcal{N}$. Fix $\phi(x, y) \in \mathcal{L}_M$ NIP. Let $p \in S^{\phi}_{U}(x)$ be invariant over M.

Define the function

$$d_p : S_M(y) \to 2$$

such that for all $t \in S_M(y)$,

$$p \vdash \phi(x, t)^{d_p(t)}.$$

- This function determines p by considering all formulae with parameters from M.

Let $\mathcal{M} \prec^+ \mathcal{N}$. Fix $\phi(x, y) \in \mathcal{L}_M$ NIP. Let $p \in S_U^{\phi}(x)$ be invariant over M.

Define the function

$$d_p : S_M(y) \to 2$$

such that for all $t \in S_M(y)$,

$$p \vdash \phi(x, t)^{d_p(t)}.$$

- This function determines p by considering all formulae with parameters from M.
- We would like to only consider ϕ^*, but in order to do so we need to look at parameters from N.

Let $\mathcal{M} \prec^{+} \mathcal{N}$. Fix $\phi(x,y) \in \mathcal{L}_M$ NIP. Let $p \in S^{\phi}_{U}(x)$ be invariant over M.

Let b, $d \in U$, $q_0 = \mathrm{tp}_N(b)$, and $q_1 = \mathrm{tp}_N(d)$.

Proposition 2.11

If both q_0 and q_1 are finitely satisfiable in M and $q_0^{\phi^*} = q_1^{\phi^*}$, then

$$p \vdash \phi(x,b) \quad \text{iff} \quad p \vdash \phi(x,d).$$

Proof: Let $\tilde{q}_i = q_i\!\upharpoonright^{U}$.

By uniqueness and existence of coheirs, $\tilde{q}_i$ is finitely satisfiable in M, so $\tilde{q}_0 \otimes \tilde{q}_1$ is finitely satisfiable in M.

Let $(b_i d_i \,:\, i < \omega) \subseteq U$ be indiscernible over M with

$$b_0 d_0 \models (\tilde{q}_0 \otimes \tilde{q}_1)\!\downharpoonleft_M .$$

Let $\mathcal{M} \prec^+ \mathcal{N}$. Fix $\phi(x, y) \in \mathcal{L}_M$ NIP. Let $p \in S^{\phi}_U(x)$ be invariant over M. Let b, $d \in U$, $q_0 = \mathrm{tp}_N(b)$, and $q_1 = \mathrm{tp}_N(d)$.

Proposition 2.11

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$$p \vdash \phi(x, b) \quad \text{iff} \quad p \vdash \phi(x, d).$$

Proof (cont.): Assume $p \vdash \phi(x, b) \wedge \neg\phi(x, d)$. Since p is invariant over M,

$$\{\phi(x, b_i) \wedge \neg\phi(x, d_i) \ : \ i < \omega\} \subseteq p,$$

so $\phi(x, b_0) \wedge \neg\phi(x, d_0)$ does not divide over M.

But Theorem 2.10 implies the existence of $a \in N$ such that

$$\tilde{q}_0 \otimes \tilde{q}_1 \vdash \phi(a, y_0) \wedge \neg\phi(a, y_1),$$

a contradiction! ■

Let $\mathcal{M} \prec^+ \mathcal{N}$. Fix $\phi(x, y) \in \mathcal{L}_M$ NIP. Let $p \in S_U^\phi(x)$ be invariant over M.

Definition

Let

$$f_p^\phi : S_N^{\phi^*}(y) \text{ fin. sat. } M \to 2$$

be given by $f_p^\phi(q) = \epsilon$ iff

$$\exists b \models q \text{ such that } \mathrm{tp}_N(b) \text{ fin. sat. } M \text{ and } p \vdash \phi(x, b)^\epsilon.$$

Let $\mathcal{M} \prec^+ \mathcal{N}$. Fix $\phi(x, y) \in \mathcal{L}_M$ NIP. Let $p \in S_U^\phi(x)$ be invariant over M.

Definition

Let

$$f_p \ : S_N^{\phi^*}(y) \text{ fin. sat. } M \to 2$$

be given by $f_p\,(q) = \epsilon$ iff

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Let $\mathcal{M} \prec^{+} \mathcal{N}$. Fix $\phi(x, y) \in \mathcal{L}_M$ NIP. Let $p \in S^{\phi}_{U}(x)$ be invariant over M.

Definition

Let

$$f_p : S^{\phi^*}_{N}(y) \text{ fin. sat. } M \to 2$$

be given by $f_p(q) = \epsilon$ iff

$$\exists b \models q \text{ such that } \mathrm{tp}_N(b) \text{ fin. sat. } M \text{ and } p \vdash \phi(x, b)^{\epsilon}.$$

- Proposition 2.11 implies that f_p is well-defined.

Let $\mathcal{M} \prec^+ \mathcal{N}$. Fix $\phi(x, y) \in \mathcal{L}_M$ NIP. Let $p \in S^{\phi}_U(x)$ be invariant over M.

Definition

Let

$$f_p : S^{\phi^*}_N(y) \text{ fin. sat. } M \to 2$$

be given by $f_p(q) = \epsilon$ iff

$$\exists b \models q \text{ such that } \operatorname{tp}_N(b) \text{ fin. sat. } M \text{ and } p \vdash \phi(x, b)^{\epsilon}.$$

- Proposition 2.11 implies that f_p is well-defined.
- Further, f_p determines p since for all $q \in S_N(y)$ fin. sat. M,

$$p \vdash \phi(x, q{\restriction}_M)^{f_p(q^{\phi^*})}.$$

Let $\mathcal{M} \prec^+ \mathcal{N}$. Fix $\phi(x, y) \in \mathcal{L}_M$ NIP. Let $p \in S_U^{\phi}(x)$ be invariant over M.

Definition

Let

$$f_p \; : S_N^{\phi^*}(y) \text{ fin. sat. } M \to 2$$

be given by $f_p\,(q) = \epsilon$ iff

$$\exists b \models q \text{ such that } \operatorname{tp}_N(b) \text{ fin. sat. } M \text{ and } p \vdash \phi(x, b)^{\epsilon}.$$

- Proposition 2.11 implies that f_p is well-defined.
- Further, f_p determines p since for all $q \in S_N(y)$ fin. sat. M,

$$p \vdash \phi(x, q\restriction_M)^{f_p(q^{\phi^*})}.$$

- Note that f_p could be defined on $S_U^{\phi^*}(y)$ fin. sat. M.

We have $d_p : S_M(y) \to 2$ such that for all $t \in S_M(y)$,

$$p \vdash \phi(x,t)^{d_p(t)},$$

and $f_p : S_N^{\phi^*}(y)$ fin. sat. $M \to 2$ such that for all $q \in S_N(y)$ fin. sat. M,

$$p \vdash \phi(x, q{\restriction}_M)^{f_p(q^{\phi^*})}.$$

$$\begin{tikzcd}
\mathcal{N} \arrow[d, dash, "+"] & S_N(y) \text{ fin. sat. } M \arrow[r, two heads, "\restriction_{\phi^*}"'] \arrow[d, two heads, "\restriction_M"] & S_N^{\phi^*}(y) \text{ fin. sat. } M \arrow[dd, "f_p"] \\
\mathcal{M} & S_M(y) \arrow[dr, "d_p"'] & \\
 & & 2
\end{tikzcd}$$

We have $d_p : S_M(y) \to 2$ such that for all $t \in S_M(y)$,

$$p \vdash \phi(x,t)^{d_p(t)},$$

and $f_p : S_N^{\phi^*}(y)$ fin. sat. $M \to 2$ such that for all $q \in S_N(y)$ fin. sat. M,

$$p \vdash \phi(x, q\restriction_M)^{f_p(q^{\phi^*})}.$$

$$\begin{tikzcd}
\mathcal{U} \arrow[d, dash, "\#"] & S_U(y) \text{ fin. sat. } M \arrow[r, two heads, "\restriction_{\phi^*}"'] \arrow[d, hook, two heads, "\restriction_N"] & S_U^{\phi^*}(y) \text{ fin. sat. } M \arrow[d, hook, two heads, "\restriction_N"] \\
\mathcal{N} \arrow[d, dash, "+"] & S_N(y) \text{ fin. sat. } M \arrow[r, two heads, "\restriction_{\phi^*}"'] \arrow[d, two heads, "\restriction_M"] & S_N^{\phi^*}(y) \text{ fin. sat. } M \arrow[dd, "f_p"] \\
\mathcal{M} & S_M(y) \arrow[dr, "d_p"'] & \\
& & 2
\end{tikzcd}$$

Let $\mathcal{M} \prec^{+} \mathcal{N} \prec^{\#} \mathcal{U}$. Fix $\phi(x,y) \in \mathcal{L}_M$ NIP.

Definition

Let

$$\Omega := 2^{S_U^{\phi^*}(y) \text{ fin. sat. } M}.$$

We view Ω in the product topology, so a subbasis for Ω is

$$\left\{ \{g \in \Omega \, : \, g(q) = \epsilon\} \, : \, q \in S_U^{\phi^*}(y) \text{ fin. sat. } M, \; \epsilon < 2 \right\}.$$

Define the injection $\quad f : S_U^{\phi}(x) \text{ inv. } M \to \Omega \quad$ by $\quad p \mapsto f_p$.

Definition

Let $\quad \mathrm{Inv}_\phi(M) := \mathrm{Image}\, f = \{f_p \, : \, p \in S_U^{\phi}(x) \text{ inv. } M\} \subseteq \Omega.$

Lemma 2.12

$\mathrm{Inv}_\phi(M)$ is closed in Ω.

Proof: For all $g \in \Omega$, define

$$\Gamma_g = \left\{ \phi(x,b)^{g(q^{\phi^*})} \ : \ q \in S_N(y) \text{ fin. sat. } M,\ b \models q \right\}.$$

Claim

If $g \in \Omega$ and $p \in S_U^\phi(x)$ inv. M, then $\quad g = f_p \quad \Leftrightarrow \quad \Gamma_g \subseteq p.$

Proof of Claim: $(\Rightarrow)$: f_p is well-defined.

$(\Leftarrow)$: Suppose $\Gamma_g \subseteq p$.

Let $q \in S_N(y)$ fin. sat. M and $b \models q$.

Then $g(q^{\phi^*}) = \epsilon$ implies $\phi(x,b)^\epsilon \in \Gamma_g \subseteq p$, so $f_p(q^{\phi^*}) = \epsilon$. $\square$

Proof of Lemma 2.12 (cont.):

Let $\mathcal{L}^* = \mathcal{L}_U \cup \{\epsilon_q : q \in S_M(y)\} \cup \{c\}$.

Let $g \in \Omega \setminus \mathrm{Inv}_\phi(M)$.

Let $\Sigma(x) = \mathrm{Th}_U(\mathcal{U}) + \{\phi(x, b) \leftrightarrow \epsilon_{\mathrm{tp}_M(b)} = c : b \in U\}$.

By the claim, $\Sigma(x) + \Gamma_g(x)$ is inconsistent.

By compactness, there is some finite $\Gamma'_g \subseteq \Gamma_g$ such that $\Sigma + \Gamma'_g$ is inconsistent.

Then there is some finite $Q \subseteq S_U^{\phi^*}(y)$ fin. sat. M such that

$$g \in \{h \in \Omega : h\restriction_Q = g\restriction_Q\} \subseteq \Omega \setminus \mathrm{Inv}_\phi(M).$$

Thus $\Omega \setminus \mathrm{Inv}_\phi(M)$ is open. ■

For all $s \in S^{\phi}_{M}(x)$, define the function

$$\hat{f}_s : S^{\phi^*}_{U}(y) \text{ fin. sat. } M \to 2$$

such that for all $q \in S_U(y)$ fin. sat. M,

$$q \vdash \phi(s, y)^{\hat{f}_s(q^{\phi^*})}.$$

Define the injection $\quad \hat{f} : S^{\phi}_{M}(x) \to \Omega \quad$ by $\quad s \mapsto \hat{f}_s$.

Claim

$\hat{f}$ induces a topology on $S^{\phi}_{M}(x)$ which is at least as fine as the standard.

Proof: Let $A = [\phi(x, b)^{\epsilon}]$ for some $b \in M$ and $\epsilon < 2$.

Recall that sets of this form are a subbasis for $S^{\phi}_{M}(x)$.

We will show that $A = \hat{f}^{-1}(\{g \in \Omega : g(q) = \epsilon\})$ where $q = \mathrm{tp}^{\phi^*}_{U}(b)$.

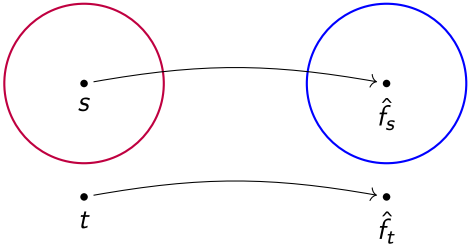
Let $q = \mathrm{tp}_U^{\phi^*}(b)$, $s \in A$, and $a \models s$.

We have $\quad \mathcal{U} \models \phi(a,b)^\epsilon \quad \Rightarrow \quad q \vdash \phi(a,y)^\epsilon \quad \Rightarrow \quad \hat{f}_s(q) = \epsilon$.

Similarly, for $t \notin A$, we have $\hat{f}_t(q) = 1 - \epsilon$. ■

$\underline{S_M^\phi}$ $\qquad\qquad$ $\underline{\Omega \;=\; 2^{S_U^{\phi^*}(y)\ \text{fin. sat. } M}}$

$A = [\phi(x,b)^\epsilon]$ $\qquad$ $B = \{g \in \Omega : g(q) = \epsilon\}$



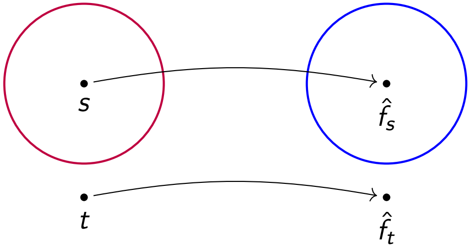
Let $q = \mathrm{tp}_{\mathcal{U}}^{\phi^*}(b)$, $s \in A$, and $a \models s$.

We have $\quad \mathcal{U} \models \phi(a,b)^\epsilon \quad \Rightarrow \quad q \vdash \phi(a,y)^\epsilon \quad \Rightarrow \quad \hat{f}_s(q) = \epsilon.$

Similarly, for $t \notin A$, we have $\hat{f}_t(q) = 1 - \epsilon$. $\blacksquare$

$\underline{S_M^\phi}$ $\qquad\qquad$ $\underline{\Omega \;=\; 2^{S_U^{\phi^*}(y)\text{ fin. sat. } M}}$

$A = [\phi(x,b)^\epsilon]$ $\qquad$ $B = \{g \in \Omega \,:\, g(q) = \epsilon\}$



Note: This demonstrates that $\hat{f}$ is injective.

The induced topology is often strictly finer.

For example:

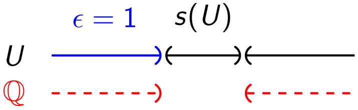
Let $\mathcal{M} = (\mathbb{Q}, <)$, $\phi(x, y)$ be $x < y$, $s(x) = \mathrm{tp}^{\phi}_{M}(\pi)$, and

$$q(y) = \left\{ (a < y)^{\exists b \in \mathbb{Q}\, [a < b < \pi]} \ : \ a \in U \right\} \in S^{\phi^*}_{U}(y).$$

Notice that q is finitely satisfiable in M. Let $B = \{g \in \Omega : g(q) = 0\}$.

By definition $q \vdash \phi(\pi, y)^{\hat{f}_s(q)}$, so $\hat{f}_s(q) = 0$. Thus $s \in \hat{f}^{-1}(B)$.

But any open neighborhood of s in $S^{\phi}_{M}(x)$ contains $\mathrm{tp}^{\phi}_{M}(c)$ for some $c \in \mathbb{Q}^{<\pi}$, and therefore the induced topology is strictly finer.



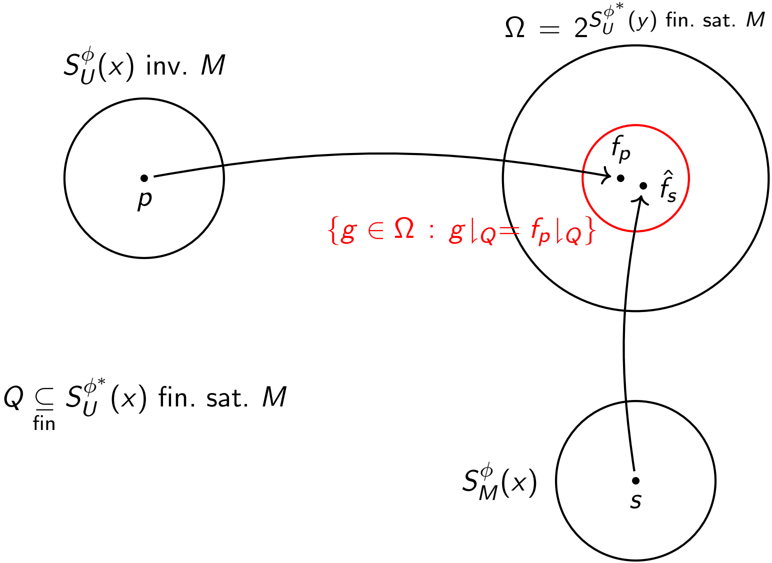
Let $\mathcal{M} \prec^{\#} \mathcal{U}$. Fix $\phi(x, y) \in \mathcal{L}_M$ NIP.

Proposition 2.13

Given $p \in S_U^{\phi}(x)$ invariant over M, f_p is in the closure of $S_M^{\phi}(x)$ when viewed in Ω, i.e.,

$$f_p \in \mathrm{cl}\left(\left\{\hat{f}_s \,:\, s \in S_M^{\phi}(x)\right\}\right).$$

Proof: For any finite $Q \subseteq S_U^{\phi^*}(y)$ fin. sat. M, we can find $s \in S_M^{\phi}(x)$ such that $f_p\restriction_Q = \hat{f}_s\restriction_Q \ldots$


$\Omega = 2^{S_U^{\phi^*}(y)}$ fin. sat. M
$S_U^{\phi}(x)$ inv. M
f_p
$\hat{f}_s$
p
$\{g \in \Omega : g\restriction_Q = f_p\restriction_Q\}$
$Q \underset{\text{fin}}{\subseteq} S_U^{\phi^*}(x)$ fin. sat. M
$S_M^{\phi}(x)$
s

Given $q_0, \ldots, q_{n-1} \in S_U(y)$ fin. sat. M, let

$$q(y_0, \ldots, y_{n-1}) = q_0(y_0) \otimes \cdots \otimes q_{n-1}(y_{n-1}),$$

and let $\bar{b} \models q{\restriction}_M$.

Let

$$\psi(x, y) := \bigwedge_{i<n} \phi(x, y_i)^{f_p(q_i^{\phi^*})}.$$

By the definition of f_p, $\psi(x, \bar{b})$ is in p, which is invariant over M, so $\psi(x, \bar{b})$ does not divide over M.

Thus, by Theorem 2.10, there is $a \in U$ such that $\psi(a, \bar{y}) \in q$.

If we let $s = \mathrm{tp}_M(a)$, then for each i,

$$f_p(q_i^{\phi^*}) = \hat{f}_s(q_i^{\phi^*}).$$

■

We can expand M so that the image of f is the closure of $S_M^\phi(x)$ when viewed in Ω.

Let $\mathcal{M} \prec^+ \mathcal{N}$. Fix $\phi(x, y) \in \mathcal{L}_M$ NIP.

Let $\mathcal{L}^* = \mathcal{L} \cup \{P_a : a \in N\}$.

Let $\mathcal{M}^*$ expand $\mathcal{M}$ such that $P_a(M) = \phi(a, M)$ for all $a \in N$.

Let $\mathcal{U}^* \succ \mathcal{M}^*$ be a monster model.
Let $\mathcal{U}$ be the $\mathcal{L}$-reduct of $\mathcal{U}^*$.
It follows that $\mathcal{U} \succ \mathcal{M}$ is a monster model.

Proposition 2.14

$$\mathrm{Inv}_\phi(M^*) = \mathrm{cl}\left(\hat{f}\left(S_M^\phi(x)\right)\right) \subseteq \Omega.$$

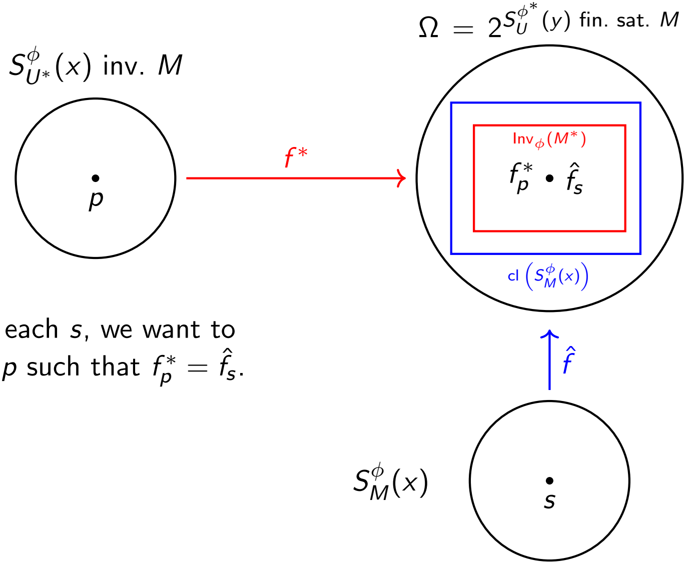
By the Coincidence Lemma:

- $S^{\phi^*}_{U^*}$ fin. sat. $M = S^{\phi^*}_{U}$ fin. sat. M
- $\Omega^* = \Omega$
- $S^{\phi}_{M^*}(x) = S^{\phi}_{M}(x)$
- $\hat{f}^* = \hat{f}$

Since $\mathrm{Aut}(U^*/M) \subseteq \mathrm{Aut}(U/M)$:

- $S^{\phi}_{U^*}(x)$ inv. $M \supseteq S^{\phi}_{U}(x)$ inv. M
- $f^* \supseteq f$
- $\mathrm{Inv}_\phi(M^*) \supseteq \mathrm{Inv}_\phi(M)$

$$\begin{tikzcd}
\mathcal{U} & \mathcal{U}^* \arrow[l, "\text{reduct}"'] \\
\mathcal{M} \arrow[r, "\text{expansion}"'] & \mathcal{M}^* \arrow[u, "\#"']
\end{tikzcd}$$



For each s, we want to find p such that $f^*_p = \hat{f}_s$.

Given $s \in S^{\phi}_{M}$, there exists $P_a(y) \in \mathcal{L}^*$ such that for all $b \in M$,

$$s \vdash \phi(x, b)^{P_a(b)}.$$

Let $p \in S^{\phi}_{U^*}(x)$ inv. M such that for all $b \in U$,

$$p \vdash \phi(x, b)^{P_a(b)}.$$

Let $q \in S_{U^*}(y)$ fin. sat. M and $\epsilon = f^*_p(q^{\phi^*})$.

Then $p \vdash \phi(x, q\restriction_M)^{\epsilon}$, and so $q \vdash P_a(y)^{\epsilon}$.

Assume $q \vdash \phi(s, y)^{1-\epsilon}$.

Since q fin. sat. M, there is $b \in M$ such that

$$\mathcal{U}^* \models P_a(b) \nleftrightarrow \phi(s, b)$$

which contradicts the definition of s.

Thus $q \vdash \phi(s, y)^{\epsilon}$, so $\hat{f}_s(q^{\phi^*}) = \epsilon$. ■

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.
Let $\phi(x, y)$, $\psi(y) \in \mathcal{L}_M$ with ϕ NIP.

Proposition 2.6

The following are equivalent:

(i) Suppose for all $b \in \psi(U)$, $\phi(x, b)$ does not divide over M.
Then there are finitely many global types such that for each $b \in \psi(U)$, $\phi(x, b)$ is in one of them. (Proposition 2.5, version 2)

(ii) Suppose $b \in U$ such that $\phi(x, b)$ does not divide over M.
Then for all global coheirs $q \in S_U(y)$ of $\mathrm{tp}_M(b)$, there is $a \in U$ such that $\phi(a, y) \in q$. (Proposition 2.5, version 3)

Proof: (ii) $\Rightarrow$ (i): Let ψ and ϕ be as in (i), and let

$$K := \{q \in S_U(y) : q \text{ finitely satisfiable in } \psi(M)\}.$$

Let $q \in K$ and $b \models q\restriction_M$. By compactness, $q + \psi$ is consistent, so $\phi(x, b)$ does not divide over M. By (ii), there is $a \in U$ such that $q \in [\phi(a, y)]$.

Since K is compact, there are $a_0, \ldots, a_{n-1} \in U$ such that the open sets $[\phi(a_i, y)]$ cover K.

Let $\bar{a}' \in U'$ be such that $\mathrm{tp}_U(\bar{a}')$ is an heir of $\mathrm{tp}_M(\bar{a})$. Let $b \in \psi(U)$.

Since $\mathrm{tp}_{M\bar{a}'}(b)$ is a coheir of $\mathrm{tp}_M(b)$, there exists $b'' \in U''$ such that $\mathrm{tp}_{U'}(b'')$ extends $\mathrm{tp}_{M\bar{a}'}(b)$ and is finitely satisfiable in M.

Then $\mathrm{tp}_U(b'') \in K$, so $\bigvee_{i<n} \phi(a_i, y) \in \mathrm{tp}_{U'}(b'')$.

M-invariance implies that

$$\bigvee_{i<n} \phi(a'_i, y) \in \mathrm{tp}_{U'}(b'') \supseteq \mathrm{tp}_{M\bar{a}'}(b),$$

and so $\mathcal{U}' \models \bigvee_{i<n} \phi(a'_i, b)$.

Let $p_i = \mathrm{tp}_U(a'_i)$. Then $\phi(x, b) \in p_i$ for some i.

(i) $\Rightarrow$ (ii): See paper. ∎

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.
Let $\phi(x, y) \in \mathcal{L}_M$ be NIP.

Lemma 2.16

The following are equivalent:

(i) Suppose $b \in U$ such that $\phi(x, b)$ does not divide over M. Then there is $\psi(y) \in \mathrm{tp}_M(b)$ such that $\{\phi(x, b) : b \in \psi(M)\}$ is consistent. (Conjecture 2.15)

(ii) Suppose $b \in U$ such that $\phi(x, b)$ does not divide over M. Then there is $a \in U$ such that for all global coheirs $q \in S_U(y)$ of $\mathrm{tp}_M(b)$, $\phi(a, y) \in q$. (Conjecture 2.15, version 2)

Proof: (i) $\Rightarrow$ (ii): By (i), there is $\psi \in \mathrm{tp}_M(b)$ and $a \in U$ such that for all $b \in \psi(M)$, we have $\mathcal{U} \models \phi(a, b)$.

(ii) $\Rightarrow$ (i): Since (i) reduces to the case where $\mathcal{L}$ and $\mathcal{M}$ are countable, we may assume $\mathcal{L}$ and $\mathcal{M}$ are countable.

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{U} \succ \mathcal{M}$ a monster model.
Let $\phi(x, y) \in \mathcal{L}_M$ be NIP.

Lemma 2.16

The following are equivalent:

(i) Suppose $b \in U$ such that $\phi(x, b)$ does not divide over M.
Then there is $\psi(y) \in \mathrm{tp}_M(b)$ such that $\{\phi(x, b) : b \in \psi(M)\}$ is consistent. (Conjecture 2.15)

(ii) Suppose $b \in U$ such that $\phi(x, b)$ does not divide over M.
Then there is $a \in U$ such that for all global coheirs $q \in S_U(y)$ of $\mathrm{tp}_M(b)$, $\phi(a, y) \in q$. (Conjecture 2.15, version 2)

Proof: (cont.) Let $b \in U$.

Suppose $\phi(x, b)$ does not divide over M, and let $a \in U$ such that for all coheirs $q \in S_U(y)$ of $\mathrm{tp}_M(b)$, $\phi(a, y) \in q$.

We claim there is $\psi \in \mathrm{tp}_M(b)$ such that for all $d \in \psi(M)$, $\mathcal{U} \models \phi(a, d)$. ■

Claim

There is $\psi \in \text{tp}_M(b)$ such that for all $d \in \psi(M)$, $\mathcal{U} \models \phi(a, d)$.

Proof of Claim:

Assume not, so for all $\psi \in \text{tp}_M(b)$ there is $d \in \psi(M) \setminus \phi(a, M)$.

Let $(\theta_i : i < \omega)$ enumerate $\text{tp}_M(b)$.

For all $i < \omega$, let $\psi_i := \bigwedge_{j<i} \theta_j$ and

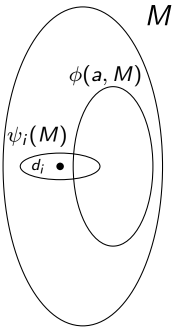
$$d_i \in \psi_i(M) \setminus \phi(a, M).$$

Thus $\text{tp}_M(d_i) \to \text{tp}_M(b)$.

Let $\mathcal{D}$ be a nonprincipal ultrafilter on ω, and let

$$q = \lim_{\mathcal{D}} \text{tp}_U(d_i) = \{\gamma \in \mathcal{L}_U : \{i : d_i \models \gamma\} \in \mathcal{D}\}.$$

Then $\phi(a, y) \notin q$ a coheir of $\text{tp}_M(b)$. $\square$



Let $\mathcal{L}$ be a language and $\mathcal{M}$ an NIP $\mathcal{L}$-structure, both countable.

Let $\phi(x, y) \in \mathcal{L}_M$.

Theorem 2.17

Suppose $b \in U$ such that $\phi(x, b)$ does not divide over M and $\mathrm{tp}_M(b)$ has only countably many global coheirs.

Then there is $a \in U$ such that for all global coheirs $q \in S_U(y)$ of $\mathrm{tp}_M(b)$, $\phi(a, y) \in q$.

Let $\mathcal{L}$ be a language and $\mathcal{M}$ an NIP $\mathcal{L}$-structure, both countable.

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Suppose $b \in U$ such that $\phi(x, b)$ does not divide over M and ~~$\mathrm{tp}_M(b)$ has only countably many global coheirs~~ the space of global coheirs of $\mathrm{tp}_M(b)$ is separable.

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Question 2.18: If $\mathcal{L}$ is a countable language and $\mathcal{M}$ is a countable pseudofinite NIP $\mathcal{L}$-structure, does every $q \in S_M(y)$ have at most countably many coheirs?